Stochastic magnetohydrodynamic turbulence in space dimensions $d \geqslant 2$

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(Received 11 April 2001; published 25 October 2001)

Interplay of kinematic and magnetic forcing in a model of a conducting fluid with randomly driven magnetohydrodynamic equations has been studied in space dimensions $d \geqslant 2$ by means of the renormalization group. A perturbative expansion scheme, parameters of which are the deviation of the spatial dimension from two and the deviation of the exponent of the powerlike correlation function of random forcing from its critical value, has been used in one-loop approximation. Additional divergences have been taken into account that arise at two dimensions and have been inconsistently treated in earlier investigations of the model. It is shown that in spite of the additional divergences, the kinetic fixed point associated with the Kolmogorov scaling regime remains stable for all space dimensions $d \geqslant 2$ for rapidly enough falling off correlations of the magnetic forcing. A scaling regime driven by thermal fluctuations of the velocity field has been identified and analyzed. The absence of a scaling regime near two dimensions driven by the fluctuations of the magnetic field has been confirmed. A renormalization scheme has been put forward and numerically investigated to interpolate between the ϵ expansion and the double expansion.

DOI: 10.1103/PhysRevE.64.056411 PACS number(s): 52.30.−q, 47.27.Gs, 11.10.Hi

I. INTRODUCTION

During the past two decades, asymptotic analysis of stochastic transport equations [Navier-Stokes equation, magnetohydrodynamic (MHD) equations, advection-diffusion equation and the like] has attracted increasing attention. Various forms of the renormalization group (RG) have proved to be particularly useful in this investigation, and a great deal of work has been carried out in the RG analysis of the stochastic Navier-Stokes equation and the problem of a passive scalar (turbulent diffusion or heat conduction) [1,2]. Somewhat less effort has been devoted to the asymptotic analysis of stochastic magnetohydrodynamics since the pioneering work of Fournier *et al.* [3] and Adzhemyan *et al.* [4]. In particular, in these papers the existence of two different anomalous scaling regimes (kinetic and magnetic) in three dimensions was established corresponding to two nontrivial infrared-stable fixed points of the renormalization group. It was also conjectured that in two dimensions, the magnetic scaling regime does not exist due to the instability of the magnetic fixed point. However, in both papers, there were flaws in the renormalization of the model in two dimensions [2,5]. Even more serious shortcomings are present in recent investigations of MHD turbulence [6,7], in which a specifically two-dimensional setup has been applied with the use of the stream function and magnetic potential. Therefore, results obtained for the two-dimensional case in these papers cannot be considered completely conclusive.

In the present paper, we have first carried out a field-theoretic RG analysis of the stochastically forced equations of magnetohydrodynamics with the proper account of additional divergences that arise in two dimensions. This gives rise to a two-parameter expansion of scaling exponents and scaling functions [5], the parameters of which are the deviation of the spatial dimension from two and the deviation of the exponent of the powerlike correlation function of random forcing from its critical value. In this double expansion, the standard procedure of minimal subtractions was used in the renormalization of the corresponding field-theoretic model. We have carried out a one-loop RG analysis of the large-scale asymptotic behavior of the model and confirmed the basic conclusions of the previous analyses [3,4] that near two dimensions a scaling regime driven by the velocity fluctuations may exist, but no magnetically driven scaling regime can occur. We have also identified a scaling regime driven by thermal fluctuations [8] of the velocity field.

Second, we have performed a renormalization of the model with a different choice of finite renormalization in order to find at which noninteger dimension the magnetic fixed point ceases to be stable. This borderline dimension was found in Ref. [3] with the use of the momentum-shell RG in a setup valid in a fixed space dimension $d>2$. In the two-parameter expansion with the deviation of the exponent of the powerlike correlation function of random forcing from its critical value ϵ and $2\delta = d-2$ as expansion parameters, this effect cannot be traced. Therefore, we have carried out a RG analysis according to the ''principle of maximum divergences'' in the sense that we have included in the renormalization all graphs relevant in two dimensions, and fixed the finite renormalization in a way that reproduces the results of a momentum-shell renormalization [9] at one-loop order. This procedure gives rise to RG functions such that in the limit of small δ, ϵ, they reproduce the results of the two-parameter expansion, and in the limit of small ϵ (but finite δ) they yield the results of the usual ϵ expansion [3,4].

We have also investigated the long-range asymptotic behavior of the model in the framework of the latter scheme without any small parameter and found, in particular, that in this case, thermal fluctuations make the value of the borderline dimension of the magnetic scaling regime significantly lower ($d_c = 2.46$) than in the ϵ expansion [3] ($d_c = 2.85$).

This paper is organized as follows: Section II starts from the functional formulation of the solution of stochastic MHD. This is convenient for the analysis based on the standard field-theoretic RG approach, the details of which are described in Sec. III. The Kolmogorov constant for MHD is calculated in Sec. IV at the leading order of the two-parameter expansion. In Sec. V, a RG analysis of the model with maximum divergences is carried out in arbitrary dimension, and the observed strong effect of thermal fluctuations is discussed. In Sec. VI, the conclusions are presented.

II. FIELD THEORY FOR STOCHASTIC MAGNETOHYDRODYNAMICS

We consider the model of stochastically forced conducting fluid described by the system of magnetohydrodynamic equations for the fluctuating velocity field $\mathbf{v}(t,\mathbf{x})\equiv\mathbf{v}(x)$ of an incompressible conducting fluid and the magnetic induction $\mathbf{B}=\sqrt{\rho\mu}\mathbf{b}$ (ρ is the density and μ the permeability of the fluid) in the form [3,4]

$$\partial_t\mathbf{v}+P[(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}-(\mathbf{b}\cdot\boldsymbol{\nabla})\mathbf{b}]-\nu_0\nabla^2\mathbf{v}=\mathbf{f}^v, \tag{2.1}$$

$$\partial_t\mathbf{b}+(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{b}-(\mathbf{b}\cdot\boldsymbol{\nabla})\mathbf{v}-\nu_0 u_0\nabla^2\mathbf{b}=\mathbf{f}^b, \tag{2.2}$$

together with the incompressibility conditions

$$\boldsymbol{\nabla}\cdot\mathbf{v}=0,\quad \boldsymbol{\nabla}\cdot\mathbf{f}^v=0,\quad \boldsymbol{\nabla}\cdot\mathbf{f}^b=0. \tag{2.3}$$

In Eq. (2.1), P is the transverse projection operator, ν_0 the (unrenormalized) kinematic viscosity, and $1/u_0$ the unrenormalized magnetic Prandtl number. In statistical applications of the field-theoretic RG, the unrenormalized (bare) parameters are the physical ones.

The statistics of $\mathbf{v}$ and $\mathbf{b}$ are completely determined by the nonlinear Eqs. (2.1),(2.2),(2.3), and the probability distribution of the external large-scale random forces $\mathbf{f}^v$, $\mathbf{f}^b$. It is customary [3,4] to consider random forces $\mathbf{f}^v$ and $\mathbf{f}^b$ having a zero-mean Gaussian distribution with correlation functions of the form

$$D_{mn}(x)=\delta(t)\int\frac{d^d\mathbf{k}}{(2\pi)^d}e^{i\mathbf{k}\cdot\mathbf{x}}D(k)\left(\delta_{mn}-\frac{k_m k_n}{k^2}\right), \tag{2.4}$$

in which the time correlations of the fields have the character of white noise, while the spatial correlations are controlled by the scalar function $D(k)$. Transversality of the matrix (2.4) is a consequence of the equations $\boldsymbol{\nabla}\cdot\mathbf{v}=\boldsymbol{\nabla}\cdot\mathbf{b}=0$.

To analyze renormalization near two dimensions, we consider Eqs. (2.1), (2.2), and (2.3) supplemented by the forcing statistics

$$\langle f_m^v(x_1)f_n^v(x_2)\rangle=u_0\nu_0^3 D_{mn}(x_1-x_2;\{1,g_{v10},g_{v20}\}),$$

$$\langle f_m^b(x_1)f_n^b(x_2)\rangle=u_0^2\nu_0^3 D_{mn}(x_1-x_2;\{a,g_{b10},g_{b20}\}),$$

$$\langle f_m^v(x_1)f_n^b(x_2)\rangle=0, \tag{2.5}$$

where

$$D_{mn}(x;\{A,B,C\})=\delta(t)\times\int\frac{d^d\mathbf{k}}{(2\pi)^d}P_{mn}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}\times[Bk^{2-2\delta-2A\epsilon}+Ck^2]. \tag{2.6}$$

All the dimensional constants g_{v10}, g_{b10}, g_{v20}, and g_{b20}, in Eq. (2.5) control the amount of randomly injected energy. The choice of the values of the parameters ϵ and δ determines the powerlike falloff of the long-range forcing correlations and the space dimension of the system under consideration.

We choose uncorrelated kinematic and magnetic driving $[\langle\mathbf{f}^v\mathbf{f}^b\rangle=0]$, because we are considering arbitrary space dimension $d\geqslant 2$ and it is not possible to define a nonvanishing correlation function of a vector field and a pseudovector field in this case. This may be done separately for integer dimensions of space, but, contrary to the claims of some authors [3,10], it is no obstacle for application of the RG [4].

The structure of the matrix D_{mn} in Eq. (2.6) reflects a more detailed intrinsic statistical definition of forcing, whose consequences are deeply discussed in Refs. [2,11]. Technically, it is necessary to accompany the long-range correlations [corresponding to the term $k^{2-2\delta-2A\epsilon}$ in Eq. (2.6)] by local correlations [described by the analytic in k^2 term in the correlation function (2.6)] in order to construct a consistent renormalization procedure for the corresponding field-theoretic model [5,12,13]. This feature has been overlooked in the previous analyses of the problem [3,4]. The prefactors $u_0\nu_0^3$ and $u_0^2\nu_0^3$ in Eq. (2.5) have been extracted for the convenience of calculations.

The definition (2.6) includes two principal—low– and high–wave-number scales—kinetic forcings separated by transition region in the vicinity of the characteristic wave number of order $O([B/C]^{1/(2\delta+2A\epsilon)})$. The forcing contribution with local correlations gives a phenomenological description of small-scale thermal fluctuations of the magnetic induction and the velocity field [8].

The long-range parts of the translational invariant correlation functions (2.6) become scale invariant at the values $\epsilon=2,a=1$. For the exponent ϵ, the value $\epsilon=2$ is physically most reasonable, since it represents the assumption that random forces in the Navier-Stokes Eq. (2.1) act at very large scales, which substitutes for the effect of boundary conditions.

We are working in an arbitrary dimension, but the renormalization will be carried out within the two-dimensional model. In two-dimensional magnetohydrodynamic turbulence, in contrast to fluid turbulence, there are direct energy cascades in both two and three dimensions. Therefore, it is natural to expect that the scaling behavior is rather similar in both cases, and we apply the same forcing spectrum in all space dimensions $d\geqslant 2$.

We use the correlation functions

$$\langle v_{j_1}(x_1)v_{j_2}(x_2)v_{j_3}(x_3)\cdots v_{j_N}(x_N)\rangle, \tag{2.7}$$

where $1\leqslant j_r\leqslant d, r=1,2,\ldots,N$ as measurable quantities for the description of turbulence statistics. We have applied the RG method to the calculation of asymptotic properties of the correlation functions in the way initiated in Ref. [14]. This approach is based on a formal mapping from the stochastic model (2.4) to a quantum-field model [15,16] with a De Dominicis-Janssen action $S\{\mathbf{v},\mathbf{v}',\mathbf{b},\mathbf{b}'\}$, which is a functional of the physical fields $\mathbf{v},\mathbf{b}$ and independent solenoidal auxiliary fields $\mathbf{v}',\mathbf{b}'$. Thus, the correlation functions (2.7) may be expressed as functional averages with the ''weight functional'' $\mathcal{W}=\exp S$. The system of the stochastic MHD Eqs. (2.1), (2.2), (2.3), (2.5), and (2.6) gives rise to the following De Dominicis-Janssen action:

$$S=\frac{1}{2}\int dx_1\int dx_2[u_0\nu_0^3 v_m'(x_1)\times D_{mn}(x_1-x_2;\{1,g_{v10},g_{v20}\})v_n'(x_2)+u_0^2\nu_0^3 b_m'(x_1)\times D_{mn}(x_1-x_2;\{a,g_{b10},g_{b20}\})b_n'(x_2)]+\int dx\{\mathbf{v}'\cdot[-\partial_t\mathbf{v}-(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}+\nu_0\nabla^2\mathbf{v}+(\mathbf{b}\cdot\boldsymbol{\nabla})\mathbf{b}]+\mathbf{b}'\cdot[-\partial_t\mathbf{b}+u_0\nu_0\nabla^2\mathbf{b}+(\mathbf{b}\cdot\boldsymbol{\nabla})\mathbf{v}-(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{b}]\}. \quad (2.8)$$

The dimensional constants g_{v10}, g_{b10}, g_{v20}, and g_{b20}, which control the amount of randomly injected energy through Eq. (2.5), play the role of expansion parameters of the perturbation theory.

III. TWO-PARAMETER EXPANSION OF THE MODEL

The action (2.8) gives rise to four three-point interaction vertices defined by the standard rules [17], and the following set of propagators:

$$\Delta_{mn}^{vv'}(\mathbf{k},t)=\Delta_{mn}^{v'v}(-\mathbf{k},-t)=\theta(t)P_{mn}(\mathbf{k})e^{-\nu_0k^2t},$$

$$\Delta_{mn}^{bb'}(\mathbf{k},t)=\Delta_{mn}^{b'b}(-\mathbf{k},-t)=\theta(t)P_{mn}(\mathbf{k})e^{-u_0\nu_0k^2t},$$

$$\Delta_{mn}^{vv}(\mathbf{k},t)=\frac{1}{2}u_0\nu_0^2P_{mn}(\mathbf{k})e^{-\nu_0k^2|t|}(g_{v10}k^{-2\epsilon-2\delta}+g_{v20}), \quad (3.1)$$

$$\Delta_{mn}^{bb}(\mathbf{k},t)=\frac{1}{2}u_0\nu_0^2P_{mn}(\mathbf{k})e^{-u_0\nu_0k^2|t|}(g_{b10}k^{-2a\epsilon-2\delta}+g_{b20})$$

in the time–wave-number representation. With due account of Galilei invariance of the action (2.8), and careful analysis of the structure of the perturbation expansion, it can be shown [4] that for any fixed space dimension $d>2$, only three one-particle irreducible (1PI) Green functions $\Gamma^{vv'}$, $\Gamma^{bb'}$, and $\Gamma^{v'bb}$ with superficial UV divergences are generated by the action. They give rise to counter terms of the form already present in the action, which thus is multiplicatively renormalizable by power counting for space dimensions $d>2$.

We would like to emphasize that the structure of renormalization should always be analyzed separately and it is not at all obvious that the nonlinear terms are not renormalized in the solution of the stochastic MHD equations. In fact, direct calculation shows that the Lorentz-force term is renormalized. There seems to be a certain amount of confusion about this point in the recent literature. For instance, in Refs. [6,10], the authors erroneously neglect renormalization of nonlinear terms as high-order effect. The approach, adopted in Ref. [6] for two-dimensional MHD turbulence, was quite recently criticized by Kim and Yang [7], who, alas, in their field-theoretic treatment of the same problem, ignore renormalization of the Lorentz force without any justification. They also neglect renormalization of the forcing correlations by effectively considering renormalization of the model at $d>2$, which does not seem to be appropriate in a setup in which the strictly two-dimensional quantities, the stream function and magnetic potential, are used for the description of the problem.

The analysis of the autocorrelation functions of the velocity field and magnetic induction is essential near two dimensions, since in two dimensions additional divergences in the graphs of the 1PI Green's functions $\Gamma^{v'v'}$ and $\Gamma^{b'b'}$ occur. The point here is [5,12] that the nonlocal term of the action (and the similar one with the auxiliary field $\mathbf{b}'$)

$$\int dt\int d^d\mathbf{x}_1\int d^d\mathbf{x}_2\mathbf{v}'(\mathbf{x}_1,t)\cdot\mathbf{v}'(\mathbf{x}_2,t)\times\int\frac{d^d\mathbf{k}}{(2\pi)^d}k^{2-2\delta-2\epsilon}e^{i\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x}_2)},$$

brought about by the force correlation functions (2.5) is not renormalized since the divergences produced by the loop integrals of the graphs are always local in space and time [17]. The simplest way to include the corresponding local counterterms $\mathbf{v}'\nabla^2\mathbf{v}'$ and $\mathbf{b}'\nabla^2\mathbf{b}'$ in the renormalization is to add corresponding *local* terms to the force correlation function at the outset in order to keep the model multiplicatively renormalizable, which is convenient technically. This is why we have used the force correlation functions (2.5) and (2.6) with both long-range and short-range correlations. As a result, the action (2.8) is multiplicatively renormalizable and allows for a standard RG asymptotic analysis [17].

In the momentum-shell analysis of Fournier *et al.* [3], these divergences were taken into account only in the special case, when the force correlation function (2.6) is local, i.e., $\propto k^2$ (formally, this was fixed by the condition $2\delta+2A\epsilon=0$, which excludes $A\epsilon$ from the parameters of the model). In the field-theoretic treatment of Adzhemyan *et al.* [4], the contribution of the additional divergences was prescribed to a renormalization of the nonanalytic term $\propto k^{2-2\delta-2\epsilon}$, although only analytic in k^2 terms are produced in the course of renormalization.

The model is regularized using a combination of analytic and dimensional regularization with the parameters ϵ and $2\delta=d-2$. As a consequence, the UV divergences appear as poles in the following linear combinations of the regularizing parameters: ϵ, δ, $2\epsilon+\delta$, and $(a+1)\epsilon+\delta$. The UV divergences may be removed by adding suitable counterterms to the basic action S_B obtained from the unrenormalized one (2.8) by the substitution of the renormalized parameters for the bare ones: $g_{v10}\to\mu^{2\epsilon}g_{v1}$, $g_{v20}\to\mu^{-2\delta}g_{v2}$, g_{b10}

$\to\mu^{2a\epsilon}g_{b1}$, $g_{b20}\to\mu^{-2\delta}g_{b2}$, $\nu_0\to\nu$, $u_0\to u$, where μ is a scale-setting parameter having the same canonical dimension as the wave number.

To construct an analog of the usual ϵ expansion with ϵ and δ as small parameters of the same order of magnitude, it is convenient to use the minimal-subtraction scheme for the renormalization. In this approach, only singular parts of the Laurent series of the superficially divergent 1PI Green's functions are included in the renormalization constants, which give rise to the counter terms added to the basic action to make the Green's functions of the resulting renormalized model UV finite. The counterterms for the basic action corresponding to the unrenormalized action (2.8) are

$$\Delta S=\int dx\Big[\nu(Z_1-1)\mathbf{v}'\nabla^2\mathbf{v}+u\nu(Z_2-1)\mathbf{b}'\nabla^2\mathbf{b}+\frac{1}{2}(1-Z_4)u\nu^3g_{v2}\mu^{-2\delta}\mathbf{v}'\nabla^2\mathbf{v}'+\frac{1}{2}(1-Z_5)u^2\nu^3g_{b2}\mu^{-2\delta}\mathbf{b}'\nabla^2\mathbf{b}'+(Z_3-1)\mathbf{v}'(\mathbf{b}\cdot\boldsymbol{\nabla})\mathbf{b}\Big], \tag{3.2}$$

where the renormalization constants Z_1,Z_2,Z_4,Z_5 renormalizing the unrenormalized (bare) parameters $e_0=\{g_{v10},g_{v20},g_{b10},g_{b20},u_0,\nu_0\}$ and the constant Z_3 renormalizing the fields $\mathbf{b}$, and $\mathbf{b}'$, are chosen to cancel the UV divergences appearing in the Green's functions constructed using the basic action. Due to the Galilean invariance of the action, the fields $\mathbf{v}'$, and $\mathbf{v}$ are not renormalized.

In a multiplicatively renormalizable model, such as Eq. (2.8), the counterterms (3.2) may be chosen in a form containing a finite number of terms of the same algebraic structure as the terms of the original action (2.8). Thus, all UV divergences of the graphs of the perturbation expansion may be eliminated by a redefinition of the parameters of the original model.

Renormalized Green's functions are expressed in terms of the renormalized parameters

$$g_{v1}=g_{v10}\mu^{-2\epsilon}Z_1^2Z_2,\quad g_{v2}=g_{v20}\mu^{2\delta}Z_1^2Z_2Z_4^{-1},$$
$$\nu=\nu_0Z_1^{-1},\quad u=u_0Z_2^{-1}Z_1, \tag{3.3}$$
$$g_{b1}=g_{b10}\mu^{-2a\epsilon}Z_1Z_2^2Z_3^{-1},\quad g_{b2}=g_{b20}\mu^{2\delta}Z_1Z_2^2Z_3^{-1}Z_5^{-1},$$

which are the parameters of the renormalized action $S_R=S_B+\Delta S$ connected with the unrenormalized action (2.8) by the relation of multiplicative renormalization

$$S_R\{\mathbf{v},\mathbf{b},\mathbf{v}',\mathbf{b}',e\}=S\{\mathbf{v},\mathbf{b}Z_3^{1/2},\mathbf{v}',\mathbf{b}'Z_3^{-1/2},e_0\},$$

where e is a shorthand for all the renormalized parameters g_{v1}, g_{v2}, g_{b1}, g_{b2}, u, and ν. Calculation of the correlation and response functions of the velocity and magnetic fields with the use of the renormalized action yields renormalized Green's functions without UV divergences.

Independence of the unrenormalized Green's functions of the scale-setting parameter μ may be expressed in the form of differential RG equations for the renormalized 1PI Green's functions. To keep notation simple, we quote these equations only for the renormalized pair-correlation functions of the velocity field and the magnetic induction. We define the Fourier transforms as

$$W^{vv}_{R\,mn}(t_1-t_2,\mathbf{k};g)=\int\frac{d^d\mathbf{x}_1}{(2\pi)^d}\langle v_m(\mathbf{x}_1,t_1)v_n(\mathbf{x}_2,t_2)\rangle e^{i\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x}_2)},$$

$$W^{bb}_{R\,mn}(t_1-t_2,\mathbf{k};g)=\int\frac{d^d\mathbf{x}_1}{(2\pi)^d}\langle b_m(\mathbf{x}_1,t_1)b_n(\mathbf{x}_2,t_2)\rangle e^{i\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x}_2)}.$$

The basic RG equations for the correlation functions are

$$\left[\mu\frac{\partial}{\partial\mu}+\beta_g\frac{\partial}{\partial g}-\gamma_1\nu\frac{\partial}{\partial\nu}\right]W^{vv}_{R\,mn}=0,$$
$$\left[\mu\frac{\partial}{\partial\mu}+\beta_g\frac{\partial}{\partial g}-\gamma_1\nu\frac{\partial}{\partial\nu}+\gamma_3\right]W^{bb}_{R\,mn}=0, \tag{3.4}$$

where $\beta_g\partial_g$ is a shorthand for

$$\beta_g\frac{\partial}{\partial g}=\beta_{gv1}\frac{\partial}{\partial g_{v1}}+\beta_{gv2}\frac{\partial}{\partial g_{v2}}+\beta_{gb1}\frac{\partial}{\partial g_{b1}}+\beta_{gb2}\frac{\partial}{\partial g_{b2}}+\beta_u\frac{\partial}{\partial u}.$$

The coefficient functions of Eqs. (3.4) β_g and γ_1 are expressed in terms of logarithmic derivatives of the renormalization constants. We use the definitions

$$\gamma_i=\mu\frac{\partial\ln Z_i}{\partial\mu}\bigg|_0,\quad \beta_g=\mu\frac{\partial g}{\partial\mu}\bigg|_0, \tag{3.5}$$

where $g=\{g_{v1},g_{v2},g_{b1},g_{b2},u\}$, and the subscript ''0'' refers to partial derivatives taken at fixed values of the bare parameters e_0. It should be noted that here the functions β_g and γ_1 are functions of the parameters g only.

Expressing the correlation functions through dimensionless scalar functions R_v and R_b as

$$W^{vv}_{R\,mn}(t,\mathbf{k};g)=\nu^2k^{-2\delta}P_{mn}(\mathbf{k})R_v(\tau,s;g),$$
$$W^{bb}_{R\,mn}(t,\mathbf{k};g)=\nu^2k^{-2\delta}P_{mn}(\mathbf{k})R_b(\tau,s;g),$$

where $s=k/\mu$, $s\in[0,1]$ is the dimensionless wave number, and $\tau=t\nu k^2$ the dimensionless time, and solving Eqs. (3.4) by the method of characteristics, we obtain the correlation functions in the form

$$W^{vv}_{R\,mn}(t,\mathbf{k};g)=P_{mn}(\mathbf{k})\overline{\nu}^2k^{-2\delta}R_v(tk^2\overline{\nu},1;\overline{g}),$$

$$W^{bb}_{R\,mn}(t,\mathbf{k};g)=e^{\int_1^s dx\ \gamma_3[\bar{g}(x)]/x}\times P_{mn}(\mathbf{k})\bar{\nu}^2 k^{-2\delta}R_b(tk^2\bar{\nu},1;\bar{g}), \quad (3.6)$$

where $\bar{g}$ is the solution of the Gell-Mann–Low equations:

$$\frac{d\bar{g}(s)}{d\ln s}=\beta_g[\bar{g}(s)], \quad (3.7)$$

and $\bar{\nu}$ is the running coefficient of viscosity

$$\bar{\nu}=\nu e^{-\int_1^s dx\ \gamma_1[\bar{g}(x)]/x}.$$

The scale-invariant asymptotic behavior of the correlation functions stems from the existence of a stable fixed point of the RG transformation $\beta_g=0$ determined by the Gell-Mann–Low Eqs. (3.7).

The definitions (3.5) and the relations (3.3) yield β functions of the form

$$\begin{aligned}\beta_{gv1}&=g_{v1}(-2\epsilon+2\gamma_1+\gamma_2),\\ \beta_{gv2}&=g_{v2}(2\delta+2\gamma_1+\gamma_2-\gamma_4),\\ \beta_{gb1}&=g_{b1}(-2a\epsilon+\gamma_1+2\gamma_2-\gamma_3),\\ \beta_{gb2}&=g_{b2}(2\delta+\gamma_1+2\gamma_2-\gamma_3-\gamma_5),\\ \beta_u&=u(\gamma_1-\gamma_2).\end{aligned} \quad (3.8)$$

At one-loop accuracy, the γ functions are

$$\begin{aligned}\gamma_1&=\frac{1}{32\pi}[u(g_{v1}+g_{v2})+g_{b1}+g_{b2}],\\ \gamma_2&=\frac{1}{8\pi}\frac{g_{v1}+g_{v2}-g_{b1}-g_{b2}}{u+1},\\ \gamma_3&=\frac{1}{16\pi}(g_{b1}+g_{b2}-g_{v1}-g_{v2}),\\ \gamma_4&=\frac{1}{32\pi}\frac{u(g_{v1}+g_{v2})^2+(g_{b1}+g_{b2})^2}{g_{v2}}.\end{aligned} \quad (3.9)$$

There are no UV divergences in the 1PI Green's function $\Gamma^{b'b'}$ in the one-loop approximation, therefore,

$$\gamma_5=0, \quad Z_5=1, \quad (3.10)$$

which is a specific property of the two-dimensional MHD.

Large-scale asymptotic behavior is governed by infrared stable fixed points of Eqs. (3.7), determined by the system of equations $\beta_g(g^*)=0$, and the conditions $\bar{g}\to g^*$, when $s\to 0$. For $\bar{g}(s)$ close to g^*, we obtain a system of linearized equations

$$\left(Is\frac{d}{ds}-\Omega\right)(\bar{g}-g^*)=0,$$

where I is the 5×5 unit matrix and the matrix $\Omega=(\partial\beta_g/\partial g)|_{g^*}$. Solutions of this system behave like $\bar{g}=g^*+\mathcal{O}(s^{\lambda_j})$, when $s\to 0$. The exponents λ_j, $j=1,2,3,4,5$ are the eigenvalues of the matrix Ω. In the vicinity of the fixed point, all the trajectories $g(s)$ approach the fixed point, if the matrix Ω is positive definite [i.e., $\mathrm{Re}(\lambda_j)>0$].

Apart from the Gaussian fixed point $g^*_{v1}=g^*_{v2}=g^*_{b1}=g^*_{b2}=0$, with no fluctuation effects on the large-scale asymptotics, which is IR stable for $\delta>0,\epsilon<0,a>0$, there are two nontrivial IR stable fixed points of the RG with non-negative g^*_{v1}, g^*_{v2}, g^*_{b1}, g^*_{b2}, and u^*.

The thermal fixed point is generated by short-range correlations of the random force with

$$g^*_{v1}=0, \quad g^*_{v2}=-4\pi(1+\sqrt{17})\delta,$$
$$g^*_{b1}=0, \quad g^*_{b2}=0, \quad (3.11)$$

and the inverse magnetic Prandtl number

$$u^*=\frac{\sqrt{17}-1}{2}\simeq 1.562. \quad (3.12)$$

Physically, the asymptotic behavior described by this fixed point is brought about by thermal fluctuations of the velocity field [8]. The region of stability of the thermal fixed point (3.11),(3.12) is $2\epsilon+3\delta<0,\delta<0$ in the δ,ϵ plane. For the magnetic forcing-decay parameter a, the stability region is determined by the inequality $8a\epsilon+(13+\sqrt{17})\delta<0$.

The *kinetic* fixed point [3] generated by the forced fluctuations of the velocity field is given by the universal inverse magnetic Prandtl number (3.12), the parameters

$$g^*_{v1}=\frac{128\pi}{9(\sqrt{17}-1)}\frac{\epsilon(2\epsilon+3\delta)}{\epsilon+\delta},$$
$$g^*_{v2}=\frac{128\pi}{9(\sqrt{17}-1)}\frac{\epsilon^2}{\delta+\epsilon}, \quad (3.13)$$

and zero couplings of the magnetic forcing

$$g^*_{b1}=g^*_{b2}=0,$$

and it may be associated with turbulent advection of the magnetic field. The values of g^*_{v1} and g^*_{v2} in Eq. (3.13) correspond to those found previously in Ref. [5]. The region of stability of the kinetic fixed point in the δ,ϵ plane is $\epsilon>0,2\epsilon+3\delta>0$. The stability of this fixed point also requires that the parameter $a<(13+\sqrt{17})/12\approx 1.427$ independent of the ratio δ/ϵ. In spite of the absence of renormalization of the forcing correlation, the momentum-shell approach [3] yields the same condition.

The system of Eqs. (3.8), (3.9), and (3.10) for the fixed points in this multicharge problem is rather complicated, and thus, has several (in general complex-number) solutions, which we do not quote explicitly here, because they are not physically relevant: apart from the fixed points listed above there are eight IR unstable real-number fixed points in the physical region (all $g \geqslant 0$) of the parameter space, and several unphysical ones. Among the unstable fixed points are, in particular, all the possible candidates to *magnetic* fixed points, i.e., fixed points with a nonvanishing magnetic coupling constant. Therefore, the conclusion made in Refs. [3,4] (although on inconsistent grounds) that the RG does not predict any magnetically driven scaling regime at and near two dimensions, is confirmed in the double-expansion approach.

It should be noted, however, that the vanishing of the function γ_5 renders the linear combinations of γ functions in the functions β_{gb1} and β_{gb2} equal: they both contain only $\gamma_1 + 2\gamma_2 - \gamma_3$ [see Eq. (3.8)]. This has the important consequence that there is no fixed point with both g_{b1} and g_{b2} nonvanishing: at least one of them must be zero. This, of course, severely reduces the set of possible magnetic fixed points at the outset.

It would be interesting, however, to follow the crossover from this regime to the scaling regime governed by the competition of the stable kinetic and magnetic fixed points, which exists in three dimensions. In the double-expansion approach, the space dimension is assumed to be close to two, therefore, the results obtained above are not applicable to this end. In the usual ϵ expansion, the leading-order value of the borderline dimension between the two regimes $d_c = (3 + \sqrt{649})/10 \approx 2.848$ [3].

All the renormalization constants and the RG functions quoted above may be calculated also at finite δ. The resulting system of fixed-point equations allows for a solution in a form of an ϵ expansion (with finite δ) and yields the same result as the usual ϵ expansion at the leading order. However, this approach is not self consistent in the sense that the field theory is not renormalizable at finite δ, but only in the form of a simultaneous expansion in the coupling constants and δ [17]. Therefore, in order to construct an interpolation procedure between the ϵ,δ expansion and the usual ϵ expansion, something has to be done with the UV divergences at $d>2$ introduced by the local terms in the force correlation functions. This issue will be dealt with in Sec. V.

IV. KOLMOGOROV CONSTANT FOR STOCHASTIC MAGNETOHYDRODYNAMICS NEAR TWO DIMENSIONS

The energy spectrum $E(k)$ in magnetohydrodynamics is given by the sum of equal-time pair-correlation functions of the velocity field and magnetic induction

$$W^{vv}_{nn}(t,\mathbf{x};t,\mathbf{x}) + W^{bb}_{nn}(t,\mathbf{x};t,\mathbf{x}) = 2\int_0^\infty dk\, E(k). \quad (4.1)$$

Note that the energy spectrum is defined through unrenormalized correlation functions. From Eqs. (3.6) and (4.1), we infer an expression for the energy spectrum in terms of the scaling functions as

$$E(k) = \frac{(d-1)k^{1-4\epsilon/3}}{(4\pi)^{d/2}\Gamma(d/2)} \left(\frac{g_{v10} u_0 \nu_0^3}{\overline{g}_{v1}\overline{u}} \right)^{2/3} [R_v(0,1;\overline{g}) + Z_3(g) e^{\int_1^s dx\, \gamma_3[\overline{g}(x)]/x} R_b(0,1;\overline{g})],$$

where for the running coefficient of viscosity $\overline{\nu}$, the expression

$$\overline{\nu} = \left(\frac{g_{v10} u_0 \nu_0^3}{\overline{g}_{v1}\overline{u}} \right)^{1/3} k^{-2\epsilon/3} \quad (4.2)$$

has been used. The relation (4.2) is a consequence of the connection between the functions β_{v1}, β_u, γ_1, and γ_2 [18].

At the kinetic fixed point, the spectrum has the form

$$E(k) = \left(\frac{g_{v10} u_0 \nu_0^3}{g^*_{v1} u^*} \right)^{2/3} \frac{(d-1)k^{1-4\epsilon/3}}{(4\pi)^{d/2}\Gamma(d/2)} [R_v(0,1;g^*) + Z_3(g) s^{\gamma_3(g^*)} R_b(0,1;g^*)]. \quad (4.3)$$

However, at leading order $R_b(0,1;g) = 1/2(g_{b1} + g_{b2})$, and since at the kinetic fixed point $g^*_{b1} = g^*_{b2} = 0$ and $\gamma_3(g^*) > 0$, only the scaling function R_v survives here.

The kinetic- and magnetic-energy injection rates ε_v and ε_b may be expressed as

$$\varepsilon_v = \frac{1}{2}\int \frac{d^d\mathbf{k}}{(2\pi)^d} \langle \mathbf{f}^v(\mathbf{k}) \cdot \mathbf{f}^v(-\mathbf{k}) \rangle,$$

$$\varepsilon_b = \frac{1}{2}\int \frac{d^d\mathbf{k}}{(2\pi)^d} \langle \mathbf{f}^b(\mathbf{k}) \cdot \mathbf{f}^b(-\mathbf{k}) \rangle. \quad (4.4)$$

For our choice of correlation functions after the introduction of simple sharp cutoffs, Eq. (4.4) yields the relation between the unrenormalized values of the coupling constants and the energy injection rates in the form

$$\varepsilon_v = \frac{(d-1)u_0 \nu_0^3}{2} \int_{k_I<k<k_d} \frac{d^d\mathbf{k}}{(2\pi)^d} (g_{v10} k^{2-2\delta-2\epsilon} + g_{v20} k^2),$$

$$\varepsilon_b = \frac{(d-1)u_0^2 \nu_0^3}{2} \int_{k_I<k<k'_d} \frac{d^d\mathbf{k}}{(2\pi)^d} \times (g_{b10} k^{2-2\delta-2a\epsilon} + g_{b20} k^2), \quad (4.5)$$

where k_I is the wave number corresponding to the integral scale, and k_d, k'_d are the characteristic wave numbers of viscous and resistive dissipation, respectively.

In the stationary state modeling developed isotropic turbulence, the energy injection is assumed to take place at large scales. Therefore, we put the parameters g_{v20} and g_{b20}, which correspond to small-scale injection of energy, equal to zero. It should be borne in mind that the corresponding running coupling constants are created in the course of renormalization regardless of the unrenormalized values of these parameters. The present perturbative calculation yields only the leading order in the ϵ,δ expansion of the amplitude coefficients in the scaling form of the correlation functions. Therefore the coupling constants should be solved from Eq. (4.5) as functions of ε_v and ε_b also only at leading order of ϵ,δ expansion. Thus, we arrive at the relations

$$\varepsilon_v = \frac{u_0 \nu_0^3 g_{v10}}{16\pi} k_d^{4-2\epsilon},$$

$$\varepsilon_b = \frac{u_0^2 \nu_0^3 g_{b10}}{16\pi} (k_d')^{4-2a\epsilon}, \tag{4.6}$$

valid for large Reynolds and magnetic Reynolds numbers, when $k_d/k_I \sim \mathrm{Re}^{3/4} \gg 1$ and $k_d'/k_I \sim \mathrm{Rm}^{3/4} \gg 1$.

Substituting the relations (4.6) in Eq. (4.3), we arrive at the spectrum in the form

$$E(k) = \frac{(u^*)^{1/3}(g_{v1}^* + g_{v2}^*)}{(2\pi)^{1/3}(g_{v1}^*)^{2/3}} \varepsilon_v^{2/3} k^{1-4\epsilon/3} k_d^{4(\epsilon-2)/3} = C_k \varepsilon_v^{2/3} k^{1-4\epsilon/3} k_d^{4(\epsilon-2)/3}$$

at the leading order of the δ,ϵ expansion. The value of the Kolmogorov constant C_k inferred from here

$$C_k = \frac{2 \times 12^{1/3} \epsilon^{1/3} (\epsilon+\delta)^{2/3}}{(2\epsilon + 3\delta)^{2/3}}$$

coincides with that obtained in the case of turbulent advection of a passive scalar [13], which is, of course, not surprising, since the magnetic field is passively advected by the velocity field in the scaling regime governed by the kinetic fixed point.

V. RENORMALIZATION WITH MAXIMUM DIVERGENCES ABOVE TWO DIMENSIONS

We want to maintain the model UV finite for $2\delta = d-2 > 0$ and simultaneously keep track of the effect of the additional divergences near two dimensions. To this end, we introduce an additional UV cutoff in all propagators, i.e., instead of the set (3.1), we use the propagators

$$\Delta_{mn}^{vv'\Lambda}(\mathbf{k},t) = \theta(t)\,\theta(\Lambda - k) P_{mn}(\mathbf{k}) e^{-\nu_0 k^2 t},$$

$$\Delta_{mn}^{bb'\Lambda}(\mathbf{k},t) = \theta(t)\,\theta(\Lambda - k) P_{mn}(\mathbf{k}) e^{-u_0 \nu_0 k^2 t},$$

$$\Delta_{mn}^{vv\Lambda}(\mathbf{k},t) = \frac{1}{2}\,\theta(\Lambda - k) u_0 \nu_0^2 P_{mn}(\mathbf{k}) e^{-\nu_0 k^2 |t|} \times (g_{v10} k^{-2\epsilon-2\delta} + g_{v20}),$$

$$\Delta_{mn}^{bb\Lambda}(\mathbf{k},t) = \frac{1}{2}\,\theta(\Lambda - k) u_0 \nu_0^2 P_{mn}(\mathbf{k}) e^{-u_0 \nu_0 k^2 |t|} \times (g_{b10} k^{-2a\epsilon-2\delta} + g_{b20}),$$

where Λ is the cutoff wave number. This change obviously does not affect the large-scale properties of the model. We would like to emphasize that the additional cutoff must be introduced uniformly in all lines in order to maintain the model multiplicatively renormalizable. An attempt to introduce the cutoff, say, in the local part of the correlation functions only by the substitution $k^2 \to \theta(\Lambda - k) k^2$ would fail to renormalize the model multiplicatively, because loop contributions to the complete (dressed) correlation function would not reproduce such a structure in the wave-vector space.

In contrast with particle field theories, we will keep the cutoff parameter Λ fixed, although large compared with the physically relevant wave-number scale. This introduces an explicit dependence on Λ in the coefficient functions of the RG, which has to be analyzed separately in the large-scale limit in the coordinate space. The setup is thus very similar to that of Polchinski [20].

The RG equations maintain the previous form (3.4), but the coefficient functions become, in general, functions of the parameters μ and Λ through the dimensionless ratio μ/Λ. Solving the RG equation by the method of characteristics, we obtain the solution

$$W_{R\,mn}^{vv}(t,\mathbf{k},\Lambda;g) = P_{mn}(\mathbf{k}) \overline{\nu}^2 k^{-2\delta} R_v\left(tk^2\overline{\nu}, 1, \frac{\mu s}{\Lambda}; \overline{g}\right),$$

$$W_{R\,mn}^{bb}(t,s\mathbf{k},\Lambda;g) = e^{\int_1^s dx\, \gamma_3(x)/x} P_{mn}(\mathbf{k}) \overline{\nu}^2 k^{-2\delta} R_b\left(tk^2\overline{\nu}, 1, \frac{\mu s}{\Lambda}; \overline{g}\right),$$

where $\overline{g}$ is now the solution of the Gell-Mann–Low equations:

$$\frac{d\overline{g}(s)}{d\ln s} = \beta_g\left[\overline{g}(s), \frac{\mu s}{\Lambda}\right],$$

with the β functions explicitly depending on s, the dimensionless wave number.

Above two dimensions, an UV renormalization of the model would require an infinite number of counterterms, and in this sense, it is not renormalizable in the limit $\Lambda \to \infty$. To avoid dealing with these formal complications, we keep the additional cutoff Λ fixed (although large), and choose the renormalization procedure according to the principle of maximum divergences [19]: the same terms of the action are renormalized as in the two-dimensional case in the previous section (3.2), but the renormalization constants may have an explicit dependence on the scale-setting parameter through the ratio μ/Λ. The two counterterms

$$\int dx [1/2(1 - Z_4) u \nu^3 g_{v2} \mu^{-2\delta} \mathbf{v}' \nabla^2 \mathbf{v}' + 1/2(1 - Z_5) u^2 \nu^3 g_{b2} \mu^{-2\delta} \mathbf{b}' \nabla^2 \mathbf{b}']$$

are superfluous in the sense that in the limit $\mu/\Lambda \to 0$, the contribution to the Green functions of the graphs containing the coupling constants g_{v2} and g_{b2} remains finite provided $2\delta = d-2$ is fixed and finite and the other counterterms are properly chosen. This is guaranteed by Polchinski's theorem [20]. We retain these counterterms in order to have a possibility to pass to the limit $\delta \to 0$ in the RG equations.

The presence of the explicit cutoff implies some technical difficulties in the calculation of the renormalization constants in the traditional field-theoretic approach, which arise because we are dealing with massless vector fields. It turns out that the coefficient functions of the RG equation are simplest in a renormalization procedure, which is similar to the momentum-shell renormalization [9]. If we were calculating over the whole wave-vector space without an explicit UV cutoff, there would not be much difference between the effort required in both approaches. The presence of the UV cutoff makes calculations with nonvanishing external wave vectors rather tedious.

Let us remember that the choice of a renormalization procedure basically is the choice of the rule according to which the counterterm contributions are extracted from the perturbation expansion of the Green's functions of the model. The usual field-theoretic prescription is as follows [21]: Consider a 1PI graph γ, let $R(\gamma)$ be the renormalized value of the graph, and let $\bar{R}(\gamma)$ the value of the graph with subtracted counterterms of all the subgraphs, then

$$R(\gamma)=\bar{R}(\gamma)-T\bar{R}(\gamma), \tag{5.1}$$

where the operator T extracts the counterterm contribution from $\bar{R}(\gamma)$. Usually, $\bar{R}(\gamma)=\gamma$ on one-loop graphs, and the renormalization scheme is specified by the action of $\bar{R}(\gamma)$ on multiloop graphs together with the definition of the operator T. The counterterms may then be constructed recursively with the use of Eq. (5.1) and the definitions of T and $\bar{R}(\gamma)$. There is a rather large freedom in the choice of the counterterm operator, but to arrive at the Green's functions finite in the limit $\Lambda\to\infty$ in two dimensions—which we want to have a connection with the double expansion—the operation $\bar{R}$ must be chosen properly.

Here, we have used a renormalization procedure, in which the operation $\bar{R}(\gamma)$ is standard [21], and the subtraction operator T is defined as follows: let $F_\gamma(\omega,\mathbf{k},\Lambda)$ be the function of external frequencies and wave vectors (which also depends on the cutoff parameter Λ) corresponding to the expression $\bar{R}(\gamma)$ (this is not a 1PI graph, in general). The subtraction operator T extracts the same set of terms of the Maclaurin-expansion in the external wave vectors, which generate the counterterms (3.2), from the *difference* $F_\gamma(\omega,\mathbf{k},\Lambda)-F_\gamma(\omega,\mathbf{k},\mu)$. These coefficients of the Maclaurin expansion are calculated at vanishing external frequencies and wave vectors. It should be noted that the coefficients of this Maclaurin expansion of the function $F_\gamma(\omega,\mathbf{k},\Lambda)$ itself may not exist in the limit $\omega\to 0$ in this ''massless'' model, but the difference $F_\gamma(\omega,\mathbf{k},\Lambda)-F_\gamma(\omega,\mathbf{k},\mu)$ allows for a Maclaurin expansion finite in the limit $\omega\to 0$ to the order required for the renormalization. The counterterm operator T and the combinatorics of the renormalization procedure for higher-order graphs may then be constructed in the standard fashion. Although this is actually not needed in the present one-loop calculation, the very possibility of this extension is necessary to guarantee that the renormalization renders the model finite in the limit $\Lambda\to\infty$ in two dimensions.

Effectively, at one-loop order, this prescription reduces the region of integration to the momentum shell $\mu<k<\Lambda$, which leads to the same calculation as in the momentum-shell renormalization. In higher orders, however, our renormalization scheme does not coincide with the momentum-shell renormalization. The point of the present renormalization procedure is that without some sort of IR cutoff, the subtraction at zero momenta and frequencies is, in general, not possible in a massless model, whereas a subtraction at vanishing frequencies and external momenta of the order of μ leads to much more complicated calculations due the heavy index structure.

At one-loop accuracy in this scheme, the γ functions are

$$\gamma_1=\frac{1}{2B}[(d^2-d-2\epsilon)u\,g_{v1}+(d^2+d-4+2a\epsilon)g_{b1}+(d^2-2)(u\,g_{v2}+g_{b2})],$$

$$\gamma_2=\frac{1}{(1+u)B}[(d-1)(d+2)(g_{v1}+g_{v2})+(d+2)(d-3)\times(g_{b1}+g_{b2})],$$

$$\gamma_3=\frac{2}{B}[g_{b1}+g_{b2}-g_{v1}-g_{v2}], \tag{5.2}$$

$$\gamma_4=\frac{d^2-2}{2g_{v2}B}[u(g_{v1}+g_{v2})^2+(g_{b1}+g_{b2})^2],$$

$$\gamma_5=\frac{2(d-2)(d+2)}{g_{b2}(1+u)B}(g_{b1}+g_{b2})(g_{v1}+g_{v2}),$$

where $B=d(d+2)\Gamma(d/2)(4\pi)^{d/2}$. These expressions reveal an additional advantage of this renormalization scheme: at one-loop order, there is no explicit dependence on μ/Λ in the coefficient functions of the RG. At one-loop level, a direct comparison with the expressions obtained in the framework of the Wilson RG is also possible: the dependence on g_{v1}, g_{b1}, and u of the β functions β_{gv1}, β_{gb1}, and β_u corresponding to Eq. (5.2) coincides with that of their counterparts of Ref. [3] up to notation.

The set of β functions generated by Eq. (5.2) allows for a fixed-point solution in the form of an ϵ expansion. Little reflection shows that the fixed-point equations in this case have a self-consistent solution with the following leading-order behavior: $u=O(1)$, $g_{v1}=O(\epsilon)$, $g_{b1}=O(\epsilon)$, $g_{v2}=O(\epsilon^2)$, and $g_{b2}=O(\epsilon^2)$. The actual fixed-point values of g_{v1}, g_{b1}, and u in the ϵ expansion, as well as the stability regions with respect to ϵ, are determined by the same set of equations as in the earlier momentum-shell [3] and field-theoretic [4] calculation above two dimensions. The stability condition with respect to the dimension of space of these fixed points is, as expected, $d>2$.

It should be noted that the function γ_5 is finite in the set (5.2), whereas in the double-expansion approach, it was equal to zero [Eq. (3.10)]. This means that magnetic fixed points with both g_{b1} and g_{b2} may exist. In fact, there is one such fixed point stable at high dimensions of space that gives rise to a magnetically driven scaling regime. This fixed point may be found in the ϵ expansion, and we have also investigated its stability numerically. Technically speaking, the appearance of a magnetic fixed point with both magnetic couplings nonvanishing would be a completely expected thing to happen in the two-loop approximation, since we have not found any symmetry reasons or the like to prevent the renor-

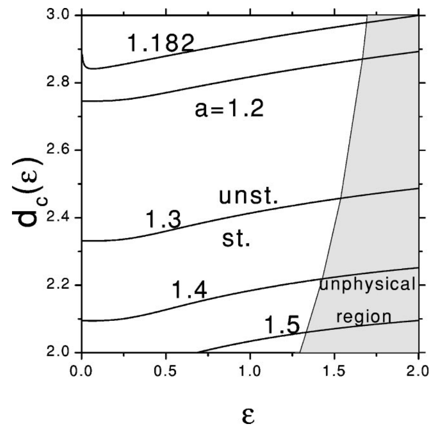

FIG. 1. The borderline dimension d_c between the stability regions of the kinetic fixed point of the RG equations (3.8) and (5.2) for magnetic forcing-decay parameter a near the double-expansion threshold $a=1.427$. This plot reveals the strong dependence of the borderline dimension on the parameter a. The shaded region on the right corresponds to values $\epsilon>2/a$, for which the forcing correlation function in the powerlike form (2.6) leads to intractable IR divergences, and a corresponding IR cutoff (magnetic integral length scale) must be introduced.

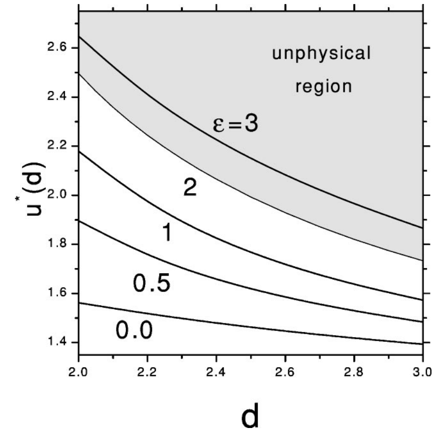

FIG. 2. The fixed-point value of the inverse magnetic Prandtl number u^* as a function of the space dimension d and the decay parameter ϵ. The lowest curve corresponds to the leading order in the ϵ expansion, which is not affected by thermal fluctuations. The shaded region in the upper part of the plot corresponds to values $\epsilon>2$, for which an IR cutoff (kinetic integral length scale) must be introduced in the correlation function (2.6).

malization of the magnetic forcing at higher orders. Thus, to investigate this effect consistently in the ϵ expansion would require a full two-loop renormalization of the model, which is beyond the scope of the present analysis.

On the other hand, in the case $\delta\sim\epsilon\rightarrow 0$, the set of coefficient functions (5.2) yields the coefficient functions (3.9) of the double expansion of the previous section. Therefore, we think that it is not totally unreasonable to use the set of coefficient functions (5.2) for an analysis of the RG fixed points for all dimensions $d\geqslant 2$.

We have investigated the stability of the kinetic fixed point given by Eq. (3.8) and (5.2) numerically for a finite range of ϵ with results depicted in Figs. 1 and 2. We think that this calculation, although it is a somewhat uncontrollable approximation, exhibits the effect of the thermal (short-range) fluctuations of the fields qualitatively correctly. The stability of the kinetic scaling regime is strongly affected by the behavior of magnetic fluctuations: from Fig. 1, it is seen that the steeper falloff of the correlations of the magnetic forcing in the wave-vector space compared with that of the kinetic forcing the lower is the space dimension, above which the kinetic fixed point is stable. In particular, when the parameter $a>1.427$, the kinetic fixed point ceases to be stable even in two dimensions. In three dimensions, the kinetic scaling regime is stable against magnetic forcing, when $a<1.15$.

The monotonic growth of the kinetic fixed point value of the inverse magnetic Prandtl number u^* as a function of the kinetic forcing-decay parameter in a fixed space dimension is depicted in Fig. 2. The plot shows also that u^* is a monotonically decreasing function of the space dimension at fixed ϵ. The lowest-lying curve corresponds to the leading order of the ϵ expansion [3]

$$u^*=\frac{1}{2}\left[-1+\sqrt{1+\frac{8(d+2)}{d}}\right].$$

We are particularly interested in the stability of the magnetic fixed point, and have carried out extensive numerical calculations of the stability of this fixed point as a function of ϵ and the space dimension d. The results are plotted in Figs. 3 and 4.

In Fig. 3, the magnetic forcing-decay parameter $a<1$ (i.e., the kinetic-forcing correlations fall off steeper in the wave-vector space than those the magnetic forcing) and it is seen that for very small a, a slowly enough decaying kinetic forcing renders the magnetic scaling regime unstable. In particular, this threshold is very small in three dimensions. With the growth of a, a strip of stability of the magnetic fixed

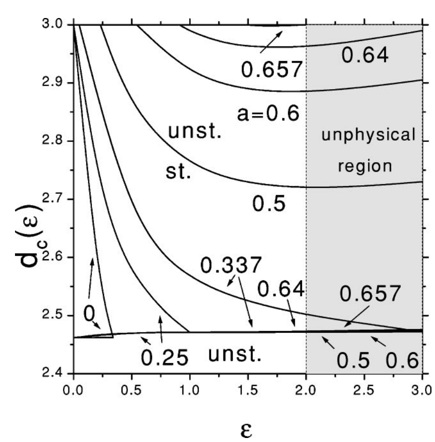

FIG. 3. The borderline dimension d_c between the stability regions of the magnetic fixed point of the RG Eqs. (3.8) and (5.2) for magnetic forcing-decay parameter $a<1$. For sufficiently small values of a, the magnetic fixed point is unstable for any finite value of ϵ, but the region of stability grows with the growth of a so that for $a>0.658$, the magnetic point becomes stable even in three dimensions for finite values of ϵ. The shading shows the region, where $\epsilon>2$, in which the powerlike correlation function (2.6) cannot be consistently used.

point in the ϵ, d plane appears such that the magnetic regime remains stable in three dimensions for all allowed kinetic forcing patterns. It is also seen that the magnetic fixed point is persistently unstable at $d \leqslant 2.46$ for all ϵ. This borderline dimension should be compared with that given by the ϵ expansion $d_c = 2.85$. From the solution, it may be seen that this significant discrepancy is due to the appearance of a stable magnetic fixed point completely different from that found in the ϵ expansion: in the latter, the magnetic fixed point is given by $g_{v1}^* = g_{v2}^* = g_{b2}^* = u^* = 0$ and $g_{b1}^* = 4d(d+2)\Gamma(d/2)(4\pi)^{d/2} a\epsilon/(d^2 - 3d - 32)$, whereas at the magnetic fixed point, whose stability is plotted in Figs. 3 and 4, only $g_{v1}^* = u^* = 0$ with nonvanishing fixed-point values of the other couplings. Thus, the lowering of the borderline dimension of stability of the magnetic scaling regime is a strong effect of the thermal fluctuations described by the short-range terms in the forcing correlation functions. Figure 4 shows the lower boundary of the stability region of the magnetic fixed point for large values of a, when magnetic-forcing correlations fall off much faster than kinetic-forcing correlations in the wave-vector space. A remarkable feature of both plots is the insensitivity of the lower border of the stability strip to the value of magnetic forcing-decay parameter a.

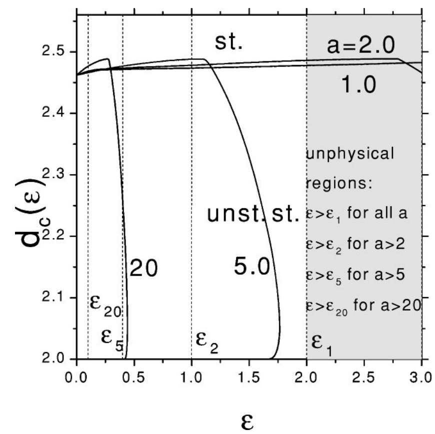

FIG. 4. The borderline dimension d_c between the stability regions of the magnetic fixed point of the RG Eqs. (3.8) and (5.2) for large values of the magnetic forcing-decay parameter $a > 1$. The shaded area and half planes with vertical dashed border lines show regions, where $\epsilon > 2/a$, in which the powerlike correlation function (2.6) cannot be consistently used.

VI. CONCLUSIONS

In conclusion, we have carried out a RG analysis of the large-scale asymptotic behavior of the solution of stochastically forced magnetohydrodynamic equations for all space dimensions $d \geqslant 2$. We have taken into account the additional divergences appearing in two dimensions ignored or improperly treated in previous work. In a two-parameter expansion scheme, we have found three infrared-stable fixed points in the physically relevant region of the parameter space spanned by the forcing parameters and the inverse magnetic Prandtl number. Anomalous scaling behavior is brought about in the basins of attraction of the thermal and kinetic fixed points, the former of which is due to thermal fluctuations, and the latter due to long-range correlated random forcing of the Navier-Stokes equation. The thermal fixed point is related to the anomalous asymptotic behavior due to thermal fluctuations near two dimensions. With a proper choice of the force-correlation function, the regime governed by the kinetic fixed point may be related to developed isotropic turbulence in a conducting fluid. We have obtained the stability condition of the kinetic fixed point near two dimensions with respect to the magnetic forcing-decay parameter as $a < 1.427$, which coincides with that of Ref. [3], but differs from the result of Ref. [4].

We have also put forward an interpolation scheme, which reproduces the earlier results in the case of an ϵ expansion in any fixed space dimension $d > 2$, and the results of the present paper in the two-parameter expansion in ϵ and $2\delta = d - 2$ near two dimensions. Using this interpolation approach, we have qualitatively analyzed the dependence of the stability of the kinetic and magnetic scaling regimes on the forcing-decay parameters and the dimension of space and found that thermal fluctuations drastically lower the borderline dimension of stability of the magnetic scaling regime.

ACKNOWLEDGMENTS

M.H. gratefully acknowledges the hospitality of the N. N. Bogoliubov Laboratory of Theoretical Physics at JINR Dubna and the Department of Physics of the University of Helsinki, Finland. J.H. thanks the Institute for Experimental Physics of the Slovak Academy of Sciences in Košice for hospitality. This paper was supported in part by Slovak Academy of Sciences (Grant No. 7232) and by the Academy of Finland (Grant No. 64935).

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